Principal Investigator: Robin Thomas
PI Institution: Georgia Institute of Technology
PI Phone Number: 404-894-2733
Contract Title: Graph Minors: Structure Theory and Algorithms
Contract Number: N00014-93-1-0325

1. Productivity measures

Refereed papers submitted but not yet published: 1
Refereed papers published: 7
Unrefereed reports and articles: 0
Books or parts thereof submitted but not yet published: 0
Books or parts thereof published: 0
Patents filed but not yet granted: 0
Patents granted: 0
Invited presentations: 3
Contributed presentations: 0
Honors received: 0
Prizes or awards received: 0
Promotions obtained: 0
Graduate students supported: 1
Post-docs supported: 0
Minorities supported: 0

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2. Summary of technical progress

2.1. Abstract. There have been significant developments in Graph Theory over the last decade that imply the existence of polynomial time algorithms for a large class of problems. These results, however, guarantee the existence of polynomial-time algorithms to solve various problems, but give no hint how to find one. Yet another drawback of these algorithms is that even though they are theoretically fast ($O(n^2)$ or $O(n^3)$), the constant hidden in the O notation is so enormous that it makes the algorithms impractical. The purpose of this project is to
  (i) further develop this theory and obtain more theoretical results,
  (ii) apply these results to the design of (at least theoretically) efficient algorithms, and
(iii) turn these theoretically efficient algorithms into practical ones.

2.2 Background. The theory of Graph Minors rests on the following two results of Robertson and Seymour. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. The first result states that in every infinite set of graphs there are two such that one is a minor of the other, and the second describes an algorithm which, for every fixed graph H decides whether an input graph has a minor isomorphic to H. When combined together these two theorems imply for instance that for every minor closed class of graphs there is a membership test that ru...s in time $O(n^3)$. However, this only proves the existence of an algorithm, but gives no clue how to construct one. Moreover, the size of the constant of proportionality makes the program impractical.

The proof of the second result of Robertson and Seymour uses the concept of tree-width of a graph. Roughly, a graph G has tree-width k if k is the minimum integer such that G can be decomposed into a "tree-structure" of pieces, each with size at most $k+1$. It can be shown that many NP-hard problems can be solved in linear time when restricted to graphs of bounded tree-width. In practice, the performamce of the algorithm depends, of course, on the bound on tree-width of the input graph. However, it turns out, quite surprisingly, that not only can this method be used for values of k up to 5, 6, or 7 (depending on the particular problem), but that there are problems in practice where the input graphs do indeed have small tree-width. Thus it is desirable to find fast algorithms to find these tree-decompositions, and that in turn requires more theoretical research into the structure of graphs and their tree-decompositions.

2.3 Hadwiger's conjecture. Hadwiger conjectured that for every positive integer p, if a graph cannot be properly p-colored, then it has a minor isomorphic to K_p. This is easy

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for $p = 1, 2, 3, 4$, and for $p = 5$ it has been shown to be equivalent to the Four Color Theorem by Wagner. With Neil Robertson and P.D.Seymour we managed to prove the next open case, that is, we have shown that the case $p = 6$ is also equivalent to the Four Color Theorem. This research was carried out prior to the commencement of the grant; a revised version of our paper was prepared during the grant period.

2.4 Rooted subdivisions of K_4. With Neil Robertson, the PI worked on the following problem: Given a graph G and four vertices of G, when does there exist a K_4-subdivision with nodes ("branch-vertices") precisely the four given vertices? Applications of this range from pure graph theory (Dirac's conjecture, Hajos' conjecture, Kelmans' conjecture) to applied problems such as efficient call routing. This is work in progress, so far we have concentrated on planar graphs.

2.5 Finding tree-decompositions of small width. My student Daniel Sanders defended his dissertation Linear time algorithms for tree-width four. He has characterized graphs of tree-width four using local operations (called "reductions"), and used his characterization to obtain a linear time algorithm to recognize graphs of tree-width at most four, and to construct an appropriate tree-decomposition for those that do have tree-width at most four.

2.6 Obstructions to tree-width four. Sanders' characterization of graphs of tree-width four gives hope for finding the excluded minors for graphs of tree-width at most four. With Bruno Courcelle we have developed a method for finding the excluded minors of a minor-closed class presented by means of "minor-closed" reductions. A modification of Sanders' original reductions yields a set of reductions that is indeed minor-closed. What remains to be done is to implement this method.

2.7 Edges in circuits. Daniel Sanders proved the following theorem. Five independent edges in a 5-connected graph are contained in a circuit, unless they form an edge cut. This is a next step toward an open conjecture of Lovasz.

2.8 Excluding the octahedron. I have studied the structure of graphs with no minor isomorphic to the octahedron. There are many similar theorems in graph theory, but this one does not seem to be known. The structure is not entirely trivial, because apart from two infinite classes (the "Möbius ladders") it also involves a lot of sporadic graphs not belonging to any infinite family (for instance, the Petersen graph, the 5-prism, and many others).

2.9 The Four Color Theorem. Since the beginning of Summer I have spent most of my time working on the Four Color Theorem. Francis Guthrie conjectured 150 years ago that every planar graph is 4-colorable; that is, its vertices can be colored using four colors in such a way that no two adjacent vertices receive the same color. A proof was given by Appel and Haken in 1976, but the argument is very complicated and to my knowledge has not yet been independently checked in its entirety.

With Neil Robertson, Dan Sanders and Paul Seymour I am currently trying to find a simpler proof that other scientists can verify. Similar to Appel and Haken's, our method proceeds in two statges. First, we establish that many configurations are "reducible" (that is, we can prove directly that they cannot be contained in a minimal counterexample to the Four Color Theorem), and then we hope to prove that in fact every planar graph contains a reducible configuration. We have found a reasonably simple "discharging" method for

proving the latter. Currently we are trying to fine-tune the discharging method, and at the same time we are enlarging our database of reducible configurations.

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3. List of publications

1. N.Robertson, P.D.Seymour and R.Thomas, A survey of linkless embeddings, Contemporary Math. 147 (1993), 125–136.

2. N.Robertson, P.D.Seymour and R.Thomas, Structural descriptions of lower ideals of trees, Contemporary Math. 147 (1993), 525–538.

3. P.D.Seymour and R.Thomas. Graph searching and a min-max theorem for tree-width. J. Combin. Theory Ser B 58 (1993), 22–33.

4. B.Oporowski, J.Oxley and R.Thomas. Typical subgraphs of 3- and 4-connected graphs. J. Combin. Theory Ser B 57 (1993). 239–257.

5. N.Robertson, P.D.Seymour and R.Thomas. Linkless embeddings of graphs in 3-space. Bull. Amer. Math. Soc. 28 (1993). 84–89.

6. P.D.Seymour and R.Thomas. Excluding infinite trees, Trans. Amer. Math. Soc. 335 (1993). 597–630.

7. N.Robertson, P.D.Seymour and R.Thomas. Hadwiger's conjecture for K_6-free graphs. Combinatorica, to appear.

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4. Transitions and DoD interactions

There have been interactions with Ohio State University and Bellcore. Bellcore scientists William Cook, Paul Seymour and Subhash Suri developed a software package for solving routing problems in graphs based on the theory of Graph Minors. A similar approach was adopted by David Appelgate (AT&T Bell Labs), Robert Bixby (Rice University), William Cook (Bellcore) and Vasek Chvatal (Rutgers University) for solving large travelling salesman problems. They used the bounded tree-width computation method to further speed up their algorithm.

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5. Software and hardware prototypes

A computer program for testing graph properties pertaining to this research area has been developed. Here are some of the functions the program can perform: deletion, contraction, addition, planarity, hamiltonicity, tree-width, path-width, connectivity, minor inclusion testing, and minor-minimality testing. No commercialization is anticipated.